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On Best Approximation by Truncated Series*

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Let T_k be the Chebyshev polynomial of the first kind of degree k. In [3] Rivlin showed that best uniform polynomial approximations to

$$f_1(x) = \sum_{j=0}^{\infty} t^j T_{aj+b}(x) = \frac{T_b(x) - tT_{|b-a|}(x)}{1 + t^2 - 2tT_a(x)}$$

are truncations, with a modification of the last term in the truncated series. That is, p_n^*, the best uniform polynomial approximation of degree n to f_1 on $[-1, +1]$, is given by

$$p_n^*(x) = \sum_{j=0}^{k} t^j T_{aj+b}(x) + \frac{t^{k+2}}{1-t^2} T_{ak+b}(x),$$

for $ak + b \leqslant n < a(k+1) + b$.

In [2] we considered

$$f_2(x) = \sum_{j=0}^{\infty} t^j U_{aj+b}(x) = \frac{U_b(x) - tU_{b-a}(x)}{1 + t^2 - 2tT_a(x)},$$

where U_k is the Chebyshev polynomial of the second kind of degree k, and we let $U_{-1}(x) = 0$ and $U_{b-a}(x) = -U_{a-b-2}(x)$ for $a > b + 1$. We attempted to obtain best uniform polynomial approximations by truncating the series and modifying two terms. This could only be done for $a = 2$. Then, for $k \geqslant 1$ and $2k + b \leqslant n < 2(k+1) + b$, the best uniform approximation of degree n to f_2 on $[-1, +1]$ is given by

$$p_n^*(x) = \sum_{j=0}^{k} t^j U_{2j+b}(x) - \frac{t^{k+2}}{(1-t)^2(1+t)} U_{2(k-1)+b}(x) - \frac{(t^2 - t - 1)}{(1-t)^2(1+t)} U_{2k+b}(x).$$

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(In the series for both f_1 and f_2, a and b are non-negative integers, $a > 0$, $-1 < t < +1$.)

We can show that f_2 differs from f_1 in the following sense. For $b_1 > 2$, there are no values of a, b_2, α, β such that $f_2(2, b_1) = \alpha f_1(a, b_2) + \beta$ for all values of t. (We have modified the notation to indicate the dependence of f_1 and f_2 on the parameters a and b.) Thus, the best approximations to $f_2(2, b_1)$ cannot be obtained by modifying the best approximations to some f_1 (a, b) solely by multiplicative and additive constants.

However, we now find that there is a simple way of deriving the approximations to $f_2(2, b_1)$.

PROPOSITION. *For* $b \geqslant 2, f_2(2,b) = (2/(1-t))f_1(2,b) + (1/(1-t))\,U_{b-2}$.

Proof. The proposition is equivalent to the equality

$$(U_b - tU_{b-2})(1-t) = 2(T_b - tT_{b-2}) + U_{b-2}(1 + t^2 - 2tT_2). \tag{1}$$

It is easy to verify (1) directly for $b = 2$ and 3. For $b \geqslant 4$, we use the identity $2T_k = U_k - U_{k-2}$ for $k = b$ and $b - 2$. Equation (1) becomes

$$2T_2 U_{b-2} = U_b + U_{b-4}. \tag{2}$$

In [1, p. 187, Eq. (36)] we have $2T_m U_{n-1} = U_{n+m-1} + U_{n-m-1}$ for $n > m$. Letting $n = b - 1$ and $m = 2$, Eq. (2) follows, and the proof is complete.

If we indicate the approximations to $f_1(2, b)$ and $f_2(2, b)$ by $p^*_{1,n}$ and $p^*_{2,n}$, respectively, then $p^*_{2,n} = (2/(1-t))\,p^*_{1,n} + (1/(1-t))\,U_{b-2}$. This relation is not obvious from simple inspection of the forms of $p^*_{2,n}$ and $p^*_{1,n}$.

Approximation by a modified truncation is a particularly useful and easy method. Clearly, this can be done for very few classes of functions. We can approximate a function which differs by a polynomial from a constant multiple of a function whose approximations are known, such as with $f_2(2, b)$ and $f_1(2, b)$, but it is not always easy to recognize that the functions are related in this manner.

Consider $f_2(a_1, b_1) = \alpha f_1(a_2, b_2) + p_m$. If this relation held for a given a_1 and b_1 and some constants α, a_2, b_2 and polynomial p_m, then we would require $a_2 = a_1$ in order to have the same poles for all values of t. We conjecture that if $a_1 \neq 2$, we cannot find suitable α, b_2, and p_m. The lack of such a relation seems to stem from the need to replace the identity $2T_k = U_k - U_{k-2}$, which was vital to the proof of the proposition for $a_1 = 2$. The inability of generating approximations to $f_2(a_1, b_1)$ for $a_1 \neq 2$ also indicates the special nature of the case $a_1 = 2$.

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